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THE MATHEMATICAL THEORY OF THE TOP

BY A. G. GREENHILL

INTEREST of mathematicians in the theory of the spinning top was stimulated by Professor Klein's *Princeton Lectures* in 1896, where he brought forward the application of his functions α, β, γ, δ, defined in terms of Euler's angles θ, ϕ, ψ by

$$\begin{aligned} \alpha &= \cos\tfrac{1}{2}\theta\, e^{\frac{1}{2}(\psi+\phi)i}, & \delta &= \cos\tfrac{1}{2}\theta\, e^{\frac{1}{2}(-\psi-\phi)i}, \\ \beta &= i\sin\tfrac{1}{2}\theta\, e^{\frac{1}{2}(\psi-\phi)i}, & \gamma &= i\sin\tfrac{1}{2}\theta\, e^{\frac{1}{2}(-\psi+\phi)i}, \end{aligned}$$

which give the complete solution in terms of multiplicative elliptic functions of the motion of any point of the symmetrical top; and the theory is receiving its complete development in his work: *Die Theorie des Kreisels*, written in collaboration with Dr. A. Sommerfeld.

In a note, communicated by Dr. Carl Barus to *Science*, Dec. 20, 1901, I submitted a method of arriving at the motion of the axis of the spinning top, applying Poinsot's principle of Angular Momentum and the Hodograph.

It is proposed to resume this treatment in the present article, and to carry on the analysis to the construction of a number of cases in which the results can be expressed in a finite algebraical form, and thus to explore the general analytical field along the easiest lines of progress, which we take to be those on which the parameter which makes its appearance of the Elliptic Integral of the Third Kind is a simple aliquot part of a period, and thereby Abel's theory can be utilized of the pseudo-elliptic integral.*

Some cases of this kind have been investigated already in articles in the *Proceedings of the London Mathematical Society*,† vols. 26 and 27, and illustrated by stereoscopic diagrams drawn by the late Mr. T. I. Dewar; but the degree given there of many of the results may be halved, so the investigation is resumed, and the analysis simplified, the notation also being selected carefully and illustrative diagrams added, drawn accurately to scale to represent an actual case of motion.

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* Abel, *Œuvres*, vol. 1, p. 104, vol. 2, p. 87.

† Abbreviated in the sequel to L. M. S.

1. The top is supposed to be symmetrical about its axis, and spinning with its point in a small, smooth cup O, like the Maxwell top;[*] as his apparatus is no longer procurable, a bicycle wheel, as in figs. 1 and 3, will be found effective for experimental demonstration.[†]

The physical constants of the top are given in C. G. S. units by

(i) the weight W in grammes (g), as weighed in a balance;

(ii) the distance h in centimetres (cm) between the point O and the centre of gravity, and then Wh (g-cm) may be called the *preponderance*;

(iii) C_1 and A_1, the moment of inertia (g-cm^2) about the axis of figure OC' and about any axis through O at right angles to OC'.

The moment of inertia A_1 can be measured experimentally by swinging the top, without revolution about OC', as a plane or conical pendulum, and observing the length l (cm) of the equivalent simple pendulum, or the angular velocity n (radians/second), or period $2\pi/n$ (seconds), when swung without rotation as a conical pendulum of small angular aperture; then

$$(1) \qquad l = \frac{A_1}{Wh}, \qquad A_1 = Whl,$$

$$(2) \qquad n^2 = \frac{g}{l} = \frac{Wgh}{A_1}.$$

2. Now spin the top about the axis OC' with angular velocity R (radians/second)—this can be done by means of a stick inserted between the spokes of the bicycle wheel—and denote the angular momentum C_1R by G' and represent it by the vector OC'; this component OC' of angular momentum G' about the axis OC' will not be altered by any motion of the axis.

This statement requires closer examination; so hold the axis OC' of the bicycle wheel in fig. 3 in any position, and then move it round in any closed path back again to its original position, previously marking one of the spokes.

If the wheel is spun with angular velocity R,

$$(1) \qquad \frac{d\phi}{dt} + \cos\theta \frac{d\psi}{dt} = R,$$

so that

$$(2) \qquad \phi + \psi = Rt + \int \text{ver}\,\theta\, d\psi,$$

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[*] *Scientific Papers*, vol. 1, p. 246.

[†] C. T. Knipp, *The Physical Review*, vol. 12, 1901.

and in a closed path the last integral is the area a cut out on the unit sphere of centre O by the axis OC'.

If the axis is held fixed again, the wheel in the absence of frictional resistance will continue spinning with angular velocity R, but the marked spoke will have acquired an angular lead a of its position if OC' had been held immovable from the first.

This is shown experimentally with great clearness when the wheel is not spun, so that $R = 0$, as in the Spherical Pendulum case.

3. If the vector OH represents the axis of resultant angular momentum H, when the point O is placed in the cup and the axis of the top is projected in any manner, OH will have components expressed in Euler's angles θ and ψ by (figs. 1 and 2)

$$(1) \qquad OC' = G', \quad C'K = A_1 \sin\theta \frac{d\psi}{dt}, \quad KH = A_1 \frac{d\theta}{dt}.$$

In accordance with the Principle of Angular Momentum the velocity of H is in the line KH parallel to the axis of the torque of gravity, and therefore perpendicular to the vertical plane $OC'K$, so that H will move horizontally and describe a curve in a fixed horizontal plane at a constant height OC above O, the vertical vector OC representing the constant component G of angular momentum about the vertical.

The velocity of H is also equal to the torque of gravity,

$$Wgh \sin\theta = A_1 n^2 \sin\theta \text{ (dyne-cm)};$$

that is, the hodograph of H, turned backwards through a right angle, is similar to the curve described by P, the projection on a horizontal plane of a point on the axis of the top, such as C', and the hodograph vector is

$$A_1 n^2 \sin\theta \, e^{\psi i}.$$

Denoting the polar coordinates of H in the horizontal plane CHK with respect to the origin C by ρ and ω, so that the vector CH is given by $\rho e^{\omega i}$, its vector velocity, turned backwards through a right angle, is

$$(2) \qquad A_1 n^2 \sin\theta \, e^{\psi i} = -i \frac{d}{dt} (\rho e^{\omega i}),$$

by which the projection P of C' is derived from the curve of H by means of a simple differentiation; and this holds also for the general unsymmetrical top, provided we can determine the curve described by H.

STEADY MOTION

4. In Steady Motion, $d\theta/dt = 0$, H and K coincide, and $CK = \rho$; denoting the constant precessional velocity $d\psi/dt$ by μ, then the gravity torque being equal to the speed of H,

$$\begin{aligned}(1)\qquad A_1 n^2 \sin\theta = \rho\mu &= (OC' \sin\theta - C'K\cos\theta)\mu \\ &= (C_1 R \sin\theta - A_1\mu \sin\theta\cos\theta)\mu;\end{aligned}$$

and dropping the factor $\sin\theta$,

$$(2)\qquad A_1\mu^2\cos\theta - C_1 R\mu + A_1 n^2 = 0,$$

$$(3)\qquad \frac{C_1 R}{A_1 n} = \frac{\mu}{n}\cos\theta + \frac{n}{\mu} = 2\sqrt{\cos\theta} + \left(\sqrt{\frac{\mu}{n}\cos\theta} - \sqrt{\frac{n}{\mu}}\right)^2,$$

a quadratic for μ when R and θ are assigned for a given top; and the reality of the roots requires the condition

$$(4)\qquad C_1 R > 2A_1 n\sqrt{\cos\theta}.$$

Interpreted geometrically in fig. 1,

$$(5)\qquad \mu = \frac{A_1 n^2 \sin\theta}{CK} = \frac{C'K}{A_1 \sin\theta},$$

$$(6)\qquad KC \cdot KC' = A_1^2 n^2 \sin^2\theta,$$

$$(7)\qquad KM \cdot KN = A_1^2 n^2;$$

or if $C'K$ cuts the vertical OC in L, and the mid-point of $C'L$ is denoted by E,

$$(8)\qquad C'E^2 - EK^2 = C'K \cdot KL = A_1^2 n^2 \sin\theta\tan\theta.$$

There is a second position of K, at K_1, and a corresponding second value μ_1 of μ, the larger root of the quadratic (2) for μ; and the critical case occurs when $EK = 0$, and

$$(9)\qquad C'E^2 = \tfrac{1}{4}C'L^2 = \tfrac{1}{4}OC'^2\tan^2\theta = A_1^2 n^2 \sin\theta\tan\theta,$$

$$(10)\qquad C_1^2 R^2 = 4A_1^2 n^2 \cos\theta, \qquad \frac{C_1 R}{A_1 n} = 2\sqrt{\cos\theta},$$

as before in (4), and now the quadratic in μ has equal roots

$$(11)\qquad \mu = n\sqrt{\sec\theta} = \frac{2A_1 n^2}{C_1 R}.$$

When the top is spun rapidly and C_1R/A_1n is great, K is close to C'; and we can take

$$(12)\qquad A_1n^2 \sin\theta = \mu \cdot KC = \mu \cdot OC' \sin\theta = C_1R\mu \sin\theta,$$

$$(13)\qquad \mu = \frac{A_1n^2}{C_1R},$$

one half the value of μ in (11); the other value μ_1 being great, K_1 being close to L.

5. To gain some idea of the effect of friction, in causing the axis of the top to rise or fall, suppose the axis of resultant angular momentum to change from OK to Ok in consequence of an impulsive couple Kk in the vertical plane KOC; and now calculate the couple Q which, together with the gravity couple $A_1n^2 \sin\theta$, is required to keep the axis OC' moving at the same inclination θ to the vertical.

Then in fig. 1,

$$(1)\qquad \frac{A_1n^2 \sin\theta + Q}{ck} = \frac{c'k}{A_1 \sin\theta},$$

$$(2)\qquad ck \cdot kc' = A_1^2n^2 \sin^2\theta + A_1Q \sin\theta = CK \cdot KC' + A_1Q \sin\theta,$$

$$(3)\qquad \frac{A_1Q}{\sin\theta} = km \cdot kn - KM \cdot KN;$$

so that Q is positive, and the axis requires to be held down in opposition to its tendency to rise, when Kk is towards the concavity of the hyperbola KK_1 having the asymptotes OC, OC'; and *vice versa*.

The resistance of the air for instance to the motion in azimuth may be considered to act as a couple with axis in the direction LC'; so that it will cause the axis to rise or fall according as OK_1 or OK is the angular momentum vector.

On the other hand the couple of resistance to spinning will have its axis in the direction $C'O$, and will cause the axis to fall in both cases.

Unsteady Motion of the Top

6. Returning to the general case of Unsteady Motion, and resolving the velocity of the vector OH in the radial direction CH,

$$(1)\qquad \frac{d\rho}{dt} = Wgh \sin\theta \cos CHK = A_1n^2 \sin\theta \frac{KH}{\rho},$$

$$(2)\qquad \rho\frac{d\rho}{dt} = Wgh \sin\theta \cdot KH = A_1^2n^2 \sin\theta \frac{d\theta}{dt},$$

and integrating,

(3) $$\tfrac{1}{2}\rho^2 = A_1^2 n^2 (E - \cos\theta),$$

where E is a constant.

This relation is also obtainable from the principle of the Conservation of Energy; the sum of the kinetic and potential energy being constant,

(4) $$\tfrac{1}{2}A_1 \frac{d\theta^2}{dt^2} + \tfrac{1}{2}A_1 \sin^2\theta \frac{d\psi^2}{dt^2} + \tfrac{1}{2}C_1R^2 + Wgh\cos\theta = K,$$

a constant; and thus

(5) $$\rho'^2 = C'H^2 = A_1^2\frac{d\theta^2}{dt^2} + A_1^2 \sin^2\theta\frac{d\psi^2}{dt^2} = 2A_1 Wgh(D - \cos\theta),$$

where

(6) $$A_1 WghD = A_1K - \tfrac{1}{2}A_1C_1R^2;$$

also

(7) $$\rho^2 - \rho'^2 = CH^2 - C'H^2 = CK^2 - C'K^2 = OC'^2 - OC^2 = G'^2 - G^2,$$

so that

(8) $$\rho^2 = 2A_1 Wgh\,(D - \cos\theta) + G'^2 - G^2 = 2A_1^2n^2\,(E - \cos\theta),$$

where

(9) $$E = D + \frac{G'^2 - G^2}{2A_1^2n^2},$$

(10) $$E + \frac{G^2}{2A_1^2n^2} = D + \frac{G'^2}{2A_1^2n^2} = F$$

suppose; and then

(11) $$OH^2 = 2A_1^2n^2\,(F - \cos\theta).$$

The quantities G, G', D, E or F may be called the dynamical constants of the problem, as contrasted with W, h, A_1, C_1, n, the physical constants.

7. To make the equations homogeneous, put

(1) $$OC = \delta, \quad OC' = \delta', \quad 4A_1 Wgh = 4A_1^2n^2 = k^2;$$

so that

(2) $$\rho^2 = \tfrac{1}{2}k^2(E - \cos\theta), \quad OH^2 = \tfrac{1}{2}k^2(F - \cos\theta),$$

and

(3) $$E + 2\frac{\delta^2}{k^2} = D + 2\frac{\delta'^2}{k^2} = F.$$

Denoting the perpendiculars CK and $C'K$ from C and C' on the tangent at H by p and p',

$$\delta' - \delta \cos\theta = p \sin\theta, \quad \delta - \delta' \cos\theta = p' \sin\theta; \tag{4}$$

also

$$\cos\theta = \frac{Ek^2 - 2\rho^2}{k^2}, \quad \sin\theta = \frac{\sqrt{[k^4 - (Ek^2 - 2\rho^2)^2]}}{k^2}, \tag{5}$$

so that, eliminating θ,

$$p^2 = \frac{[\delta' k^2 - \delta(Ek^2 - 2\rho^2)]^2}{k^4 - (Ek^2 - 2\rho^2)^2}. \tag{6}$$

Otherwise,

$$\cos\theta = E - 2\frac{\rho^2}{k^2} = \frac{\delta\delta' - pp'}{OK^2}, \tag{7}$$

with

$$OK^2 = p^2 + \delta^2 = p'^2 + \delta'^2; \tag{8}$$

so that, eliminating p',

$$\left[(p^2 + \delta^2)\left(E - 2\frac{\rho^2}{k^2}\right) - \delta\delta'\right]^2 = p^2 p'^2 = p^2(p^2 + \delta^2 - \delta'^2), \tag{9}$$

and dividing out a factor $p^2 + \delta^2$,

$$(p^2 + \delta^2)\left(E - 2\frac{\rho^2}{k^2}\right)^2 - 2\delta\delta'\left(E - 2\frac{\rho^2}{k^2}\right) - p^2 + \delta'^2 = 0, \tag{10}$$

$$p^2 = \frac{\left[\delta' - \delta\left(E - 2\frac{\rho^2}{k^2}\right)\right]^2}{1 - \left(E - 2\frac{\rho^2}{k^2}\right)^2}, \tag{11}$$

as before; or

$$OK^2 = \frac{\delta^2 + \delta'^2 - 2\delta\delta'\left(F - 2\frac{OH^2}{k^2}\right)}{1 - \left(F - 2\frac{OH^2}{k^2}\right)^2}. \tag{12}$$

This is found to represent the characteristic property of a Poinsot herpolhode, the curve produced by rolling a quadric surface about its centre, which is fixed, on a fixed tangent plane (the *invariable plane*); as for instance in the motion of a body tossed in the air, when the momental ellipsoid at the centre of gravity rolls about the C. G. on a plane at a fixed distance from the C. G., the plane remaining perpendicular to the axis of resultant angular momentum of the body; this is shown experimentally in Dr. Hermann Grassmann's *Modelle zur Kreiseltheorie*, constructed by Martin Schilling, Halle a. S.

In the terminology of Routh's *Rigid Dynamics*, vol. 2, §175, OC is the invariable line, and OC' the conjugate line.

The same equation (6) or (12) defines also the trace of a rolling line of curvature, the intersection of a concentric ellipsoid and confocal hyperboloid of two sheets, and then θ will be the variable angle between the generating lines through H of the deformable confocal hyperboloid of one sheet.*

The same curve is also the projection of a tortuous Elastica, as a consequence of Kirchhoff's *Kinetic Analogue*; and the spherical curve of C' is a hodograph of the Elastica, described with constant velocity.

8. Then

(1) $$KH^2 = \rho^2 - p^2 = \frac{R}{k^4 - (Ek^2 - 2\rho^2)^2},$$

where

(2) $$R = \rho^2[k^4 - (Ek^2 - 2\rho^2)^2] - [\delta' k^2 - \delta(Ek^2 - 2\rho^2)]^2,$$
$$= 4(\rho_1^2 - \rho^2)(\rho_2^2 - \rho^2)(\rho_3^2 - \rho^2)$$

suppose, with

(3) $$\rho_1^2 < 0 < \rho_2^2 < \rho^2 < \rho_3^2;$$

and now

(4) $$\frac{d\rho^2}{dt} = 2A_1 n^2 \sin\theta \cdot KH = \frac{n}{k}\sqrt{R},$$

(5) $$nt = \int_{\rho_2}^{\rho} \frac{k\,d\rho^2}{\sqrt{R}},$$

an elliptic integral of the first kind.

Also, denoting the angle KCH by χ,

(6) $$\tan\chi = \frac{\sqrt{(\rho^2 - p^2)}}{p} = \frac{\sqrt{R}}{\delta' k^2 - \delta(Ek^2 - 2\rho^2)},$$

(7) $$\sin\theta\, e^{\chi i} = \frac{\delta' k^2 - \delta(Ek^2 - 2\rho^2) + i\sqrt{R}}{k^2\rho}.$$

Since

(8) $$KH^2 = CH^2 - CK^2,$$

(9) $$A_1^2 \frac{d\theta^2}{dt^2} = 2A_1^2 n^2 (E - \cos\theta) - \frac{(G' - G\cos\theta)^2}{\sin^2\theta};$$

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* *Cf.* Darboux in Despeyrous' *Cours de mécanique*, vol. 2, p 522.

and putting $\cos\theta = z$,

$$\frac{dz^2}{dt^2} = 2n^2 Z, \tag{10}$$

where

$$Z = (E - z)(1 - z^2) - \tfrac{1}{2}\left(\frac{G' - Gz}{A_1 n}\right)^2 \tag{11}$$

$$= (E - z)(1 - z^2) - 2\left(\frac{\delta' - \delta z}{k}\right)^2.$$

Denoting the roots of $Z = 0$ by z_1, z_2, z_3,

$$z_1 + z_2 + z_3 = F, \tag{12}$$

and arranged so that

$$z_1 > 1 > z_2 > z > z_3 > -1, \tag{13}$$

then with $d\theta/dt$ positive and dz/dt negative, as in fig. 1,

$$nt = \int_z^{z_2} \frac{dz}{\sqrt{(2Z)}}, \tag{14}$$

an elliptic integral of the first kind, reducible to Legendre's standard form by the substitution

$$z = z_2 \sin^2\varphi + z_3 \cos^2\varphi, \tag{15}$$

so that

$$z_1 - z = (z_1 - z_3)\Delta^2\varphi = (z_1 - z_3)(1 - \kappa^2 \sin^2\varphi), \tag{16}$$

$$z_2 - z = (z_2 - z_3)\cos^2\varphi, \qquad z - z_3 = (z_2 - z_3)\sin^2\varphi$$

$$\kappa^2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad \kappa'^2 = \frac{z_1 - z_2}{z_1 - z_3},$$

and

$$nt = \sqrt{\frac{2}{z_1 - z_3}} \int_\varphi^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = \sqrt{\frac{2}{z_1 - z_3}}\left(K - F\varphi\right), \tag{17}$$

$$F\varphi = K - mt, \text{ where } \frac{m}{n} = \sqrt{\frac{z_1 - z_3}{2}}\,. \tag{18}$$

Then in Jacobi's notation

$$\varphi = \operatorname{am}(K - mt), \tag{19}$$

and, in Gudermann's notation,

$$z = z_2 \operatorname{sn}^2(K - mt) + z_3 \operatorname{cn}^2(K - mt), \tag{20}$$

and so also

$$\rho^2 = \rho_2^2 \operatorname{sn}^2(K - mt) + \rho_3^2 \operatorname{cn}^2(K - mt). \tag{21}$$

Interpreted dynamically, equation (20) shows that the axis of the top keeps time with the beats of a simple pendulum of length

(22) $$L = \frac{l}{\frac{1}{2}(z_1 - z_3)},$$

suspended from a point at a height $\frac{1}{2}(z_1 + z_3)l$ vertically above O, in such a manner that a point on the pendulum at a distance

(23) $$x = \tfrac{1}{2}(z_1 - z_3)l = \frac{l^2}{L}$$

from its point of suspension keeps at the same level as the point on the axis of the top at a distance l from O, the point of support of the top.

9. Next resolving transversely to CH, or taking the moment of the velocity of H round C,

(1) $$\rho^2 \frac{d\omega}{dt} = A_1 n^2 \sin\theta \cdot CK = A_1 n^2 (G' - G\cos\theta)$$
$$= \frac{G\rho^2}{2A_1} - A_1 n^2 (GE - G'),$$

so that

(2) $$\frac{d\omega}{dt} = \frac{G}{2A_1} - \frac{A_1 n^2 (GE - G')}{\rho^2},$$

(3) $$\omega = \frac{Gt}{2A_1} - A_1 n (GE - G') \int_0^t \frac{n\,dt}{\rho^2} = \frac{\delta}{k} nt - \tfrac{1}{2}\int_{\rho_2}^{\rho} \frac{(\delta E - \delta')k^2 d\rho^2}{\rho^2 \sqrt{R}},$$

involving an elliptic integral of the third kind.

The reduction to the standard form is effected by putting

(4) $$s - s_a = \tfrac{1}{2}M^2 (z - z_a) = M^2 \frac{\rho_a^2 - \rho^2}{k^2}, \qquad a = 1, 2, 3,$$

where M is a homogeneity factor at our disposal; and now from (5) or (14) in §8,

(5) $$nt = \int_s^{s_2} \frac{M\,ds}{\sqrt{S}}, \qquad S = 4(s - s_1)(s - s_2)(s - s_3);$$

and Weierstrass's elliptic argument u is given by

(6) $$u = \int_s^\infty \frac{ds}{\sqrt{S}} = \int_{s_2}^\infty + \int_s^{s_2} = \omega_2 + \frac{nt}{M},$$

so that s is an elliptic function of u which we can denote by $s(u)$, differing from Weierstrass's function $\wp u$ by a constant, while $\sqrt{S}$ is $-\wp' u$.

If v, σ or $s(v)$, Σ or $S(v)$ denote the values of u, s or $s(u)$, S or $S(u)$ corresponding to $z = E$, $\rho^2 = 0$,

$$(7) \qquad \wp v - \wp u = \sigma - s = \tfrac{1}{2} M^2 (E - z) = M^2 \frac{\rho^2}{k^2},$$

$$(8) \qquad \wp'^2 u = S = \tfrac{1}{2} M^6 Z = M^6 \frac{R}{k^6};$$

$$(9) \qquad -i\wp' v = \sqrt{-\Sigma} = M^3 \frac{\delta E - \delta'}{k};$$

and the order of arrangement is

$$(10) \qquad z_1 > E > z_2 > z > z_3,$$

$$(11) \qquad \rho_1^2 < 0 < \rho_2^2 < \rho^2 < \rho_3^2;$$

$$(12) \qquad s_1 > \sigma > s_2 > s > s_3;$$

so that, f denoting a real fraction,

$$(13) \qquad v = \omega_1 + f\omega_3, \text{ or } K + fK'i,$$

as it may be written when the argument u or v is supposed to be normalized into Legendre's form.

We may denote $\sqrt{-\Sigma}$ by $2Q(v)$ and now (3) becomes

$$(14) \qquad \omega = \frac{\delta}{k} nt - \int_s^{s_2} \frac{Q(v)\,ds}{(\sigma - s)\sqrt{S}}.$$

10. We choose as the standard form of the corresponding elliptic integral of the third kind,

$$(1) \qquad I(v) = \int_s^{s_2} \frac{-P(v)(\sigma - s) + Q(v)}{\sigma - s} \frac{ds}{\sqrt{S}},$$

where $P(v)$ is a function chosen so as to make

$$(2) \qquad I(v) = \tfrac{1}{2}\, i \log \frac{\theta(u + v)}{\theta(u - v)},$$

when u and v are normalized to Legendre's form; and thus

$$(3) \qquad iP(v) = \frac{\eta_3}{\omega_3} v - \zeta v,$$

and in terms of Jacobi's Zeta function,

$$\frac{P(v)}{\sqrt{(s_1 - s_3)}} = \mathrm{Z}(fK', \kappa') = \mathrm{zn} fK', \tag{4}$$

in Glaisher's notation.

Now

$$\omega = \frac{\delta}{k} nt - P(v) \int_{s}^{s_2} \frac{ds}{\sqrt{S}} - I(v), \tag{5}$$

$$I(v) = pt - \omega, \tag{6}$$

where

$$\frac{p}{n} = \frac{\delta}{k} - \frac{P(v)}{M}. \tag{7}$$

It is convenient to write I, P, Q in the sequel for $I(v)$, $P(v)$, $Q(v)$; also to put

$$x = \sigma - s = \tfrac{1}{2} M^2 (E - z) = M^2 \frac{\rho^2}{k^2}, \tag{8}$$

and, following Darboux, to put

$$\frac{\delta}{k} = \frac{L}{M}, \quad \frac{\delta'}{k} = \frac{L'}{M}, \tag{9}$$

$$Z = (E - z)(1 - z^2) - 2\left(\frac{L' - Lz}{M}\right)^2; \tag{10}$$

so that

$$nt = \int_{x_2}^{x} \frac{M dx}{\sqrt{X}}, \quad pt = (L - P) \int_{x_2}^{x} \frac{dx}{\sqrt{X}}, \tag{11}$$

$$\begin{aligned} X &= 4(x_3 - x)(x - x_2)(x - x_1), \\ &= 4(-x^3 + Ax^2 - Bx - Q^2) = S = \tfrac{1}{2} M^6 Z, \end{aligned} \tag{12}$$

$$x_1 < 0 < x_2 < x < x_3; \tag{13}$$

$$\begin{aligned} x_3 - x &= (x_3 - x_2)\mathrm{sn}^2(K - mt), \\ x - x_2 &= (x_3 - x_2)\mathrm{cn}^2(K - mt), \\ x - x_1 &= (x_3 - x_1)\mathrm{dn}^2(K - mt), \\ mt &= \sqrt{x_3 - x_1}\, \frac{nt}{M}, \end{aligned} \tag{14}$$

$$I = \int_{x_2}^{x} \frac{-Px + Q}{x} \frac{dx}{\sqrt{X}}. \tag{15}$$

Now, with

$$(16)\qquad z_1 + z_2 + z_3 = F = E + 2\,\frac{L^2}{M^2} = D + 2\,\frac{L'^2}{M^2},$$

$$(17)\qquad 3\wp v = x_1 + x_2 + x_3 = A = \tfrac{1}{2}M^2(3E - F) = M^2E - L^2,$$

$$(18)\qquad M^2E = L^2 + A,$$

$$(19)\qquad M^2F = 3(L^2 + \wp v),$$

$$(20)\qquad M^2z = M^2E - 2x = L^2 + A - 2x,$$

$$(21)\qquad -i\wp' v = \sqrt{(-4x_1x_2x_3)} = 2Q = M^3\left(E\frac{\delta}{k} - \frac{\delta'}{k}\right)$$

$$= M^3\left(E\frac{L}{M} - \frac{L'}{M}\right) = M^2EL - M^2L',$$

$$(22)\qquad M^2L' = M^2EL - 2Q = L^3 + AL - 2Q = L^3 + 3L\wp v + i\wp' v,$$

$$(23)\qquad 2\wp'' v = 4(x_2x_3 + x_3x_1 + x_1x_2) = 4B = M^4(3E^2 - 2EF + z_2z_3 + z_3z_1 + z_1z_2)$$

$$= M^4\left(E^2 - 4E\,\frac{L^2}{M^2} + 4\,\frac{L'}{M^2} - 1\right)$$

$$= M^4(E^2 - 1) - 4LM^2(LE - L') = M^4(E^2 - 1) - 8LQ,$$

so that

$$(24)\qquad M^4 = (L^2 + A)^2 - 8LQ - 4B,$$

thus determining the homogeneity factor M when the data are L and $\sigma = s(v)$; and now

$$(25)\qquad \frac{G}{2A_1n} = \frac{\delta}{k} = \frac{L}{M},$$

$$(26)\qquad \frac{G'}{2A_1n} = \frac{\delta'}{k} = \frac{L'}{M} = \frac{L^3 + AL - 2Q}{M^3}.$$

11. The angle χ, defined in (6) §8, is also given by

$$(1)\qquad \tan\chi = \frac{KH}{CK} = \frac{A_1\sin\theta\dfrac{d\theta}{dt}}{CK\sin\theta} = \frac{A_1n\sqrt{2Z}}{G' - Gz},$$

$$(2)\qquad \sin\theta\, e^{\chi i} = \frac{2\,\dfrac{L' - Lz}{M} + i\sqrt{2Z}}{\sqrt{[2(E - z)]}},$$

and

(3) $$\frac{L' - Lz}{M} = 2\,\frac{Lx - Q}{M^3},$$

(4) $$2Z = 4X/M^6,$$

(5) $$2(E - z) = 4x/M^2,$$

so that

(6) $$\tfrac{1}{2}M^2 \sin\theta\, e^{\chi i} = \frac{Lx - Q + i\sqrt{\tfrac{1}{4}X}}{\sqrt{x}},$$

and

(7) $$(\tfrac{1}{2}M^2 \sin\theta)^2 = \frac{(Lx - Q)^2 - x^3 + Ax^2 - Bx - Q^2}{x}$$
$$= -\,2LQ - B + (L^2 + A)x - x^2.$$

Combined with the relation

(8) $$(\tfrac{1}{2}M^2 z)^2 = [\tfrac{1}{2}(L^2 + A) - x]^2,$$

we obtain by addition

(9) $$\tfrac{1}{4}M^4 = \tfrac{1}{4}(L^2 + A)^2 - 2LQ - B$$

as before, in (24) §10; and this can be written

(10) $$M^4 = (L^2 - x_1 + x_2 + x_3)^2 - 4(L\sqrt{-x_1} + \sqrt{x_2 x_3})^2$$
$$= (L^2 + x_1 - x_2 + x_3)^2 + 4(L\sqrt{x_2} - \sqrt{-x_1 x_3})^2$$
$$= (L^2 + x_1 + x_2 - x_3)^2 + 4(L\sqrt{x_3} - \sqrt{-x_1 x_2})^2.$$

12. Putting

(1) $$z_1 = \operatorname{ch}\theta_1, \quad z_2 = \cos\theta_2, \quad z_3 = \cos\theta_3, \quad \theta_2 < \theta < \theta_3;$$

(2) $$M^2 \operatorname{ch}\theta_1 = L^2 - x_1 + x_2 + x_3,$$

so that

(3) $$M^4 \operatorname{sh}^2\theta_1 = (L^2 - x_1 + x_2 + x_3)^2 - M^4 = 4\,(L\sqrt{-x_1} + \sqrt{x_2 x_3})^2,$$

(4) $$\tfrac{1}{2}M^2 \operatorname{sh}\theta_1 = L\sqrt{-x_1} + \sqrt{x_2 x_3};$$

and similarly

$$\tfrac{1}{2}M^2\sin\theta_2 = L\sqrt{x_2} - \sqrt{-x_1x_3}, \tag{5}$$

$$\tfrac{1}{2}M^2\sin\theta_3 = L\sqrt{x_3} - \sqrt{-x_1x_2}, \tag{6}$$

$$M^2\cos\theta_2 = L^2 + x_1 - x_2 + x_3, \tag{7}$$

$$M^2\cos\theta_3 = L^2 + x_1 + x_2 - x_3. \tag{8}$$

13. Now, with

$$\frac{\sqrt{x}}{M} = \frac{\rho}{k}, \tag{1}$$

$$\chi = \omega - \psi, \quad I = pt - \omega, \tag{2}$$

we obtain

$$\tfrac{1}{2}M^2\sin\theta\, e^{(pt-\psi)i} = \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{\sqrt{x}}\, e^{iI} \tag{3}$$

$$= \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}\,\frac{\rho}{k}\, e^{(pt-\omega)i},$$

and thus the vector of the horizontal projection P of a point C' on the axis of the top is determined, when once the curve of H is known, either as its hodograph, or more expeditiously by means of equation (3) ; and expressed analytically, the curve of H is given by

$$\frac{\rho^2}{k^2} = \frac{\wp v - \wp u}{M^2} = \frac{\sigma(u+v)\sigma(u-v)}{M^2\sigma^2u\,\sigma^2v}, \tag{4}$$

or normalized in the Jacobian form

$$\frac{\rho^2}{k^2} = \frac{\theta^2(0)\theta(u+v)\theta(u-v)}{\lambda^2\theta^2u\theta^2v}, \tag{5}$$

while

$$e^{2Ii} = \frac{\theta(u-v)}{\theta(u+v)}, \qquad I = pt - \omega, \tag{6}$$

so that

$$\frac{\rho}{k}\, e^{(pt-\omega)i} = \frac{\theta(0)\theta(u-v)}{\lambda\theta u\theta v}, \tag{7}$$

and the right hand side is an algebraical function when v is an aliquot part of a period.

14. We arrive at the same result by the method of the hodograph of equation (2) §3; changing the sign of i,

$$(1)\qquad A_1 n^2 \sin\theta\, e^{-\psi i} = i\frac{d}{dt}(\rho e^{-\omega i}) = \rho e^{-\omega i}\left(\frac{d\omega}{dt} + \frac{i}{\rho}\frac{d\rho}{dt}\right);$$

which with

$$(2)\qquad \frac{\rho^2}{k^2} = \frac{x}{M^2},\quad \frac{i}{\rho}\frac{d\rho}{dt} = \frac{1}{2}\frac{i}{x}\frac{dx}{dt} = \frac{n}{M}\frac{\frac{1}{2}i\sqrt{X}}{x},$$

$$(3)\qquad \frac{d\omega}{dt} = p - \frac{dI}{dx}\frac{dx}{dt} = \frac{L-P}{M}n - \frac{-Px+Q}{x}\frac{n}{M} = \frac{Lx-Q}{x}\frac{n}{M},$$

leads to

$$(4)\qquad \tfrac{1}{2}M^2 \sin\theta\, e^{-\psi i} = \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}\frac{\rho}{k}e^{-\omega i},$$

as before in (3) § 13; and thus the vector of H is turned into the vector of P, the projection of C', by the factor

$$(5)\qquad \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}.$$

15. Before proceeding to the discussion of the algebraical cases as the chief object of this paper, we can examine some special results of the general problem.

When the axis passes periodically through the highest or lowest vertical position, the *rosette* curves are produced, described by Professor Klein in the *Bulletin of the American Mathematical Society*, vol. 3.

In such cases

$$(1)\qquad L' = \pm L,\quad M^4 L'^2 = M^4 L^2,\quad D = E,$$

$$(2)\qquad (L^3 + AL - 2Q)^2 = L^2(L^2 + A)^2 - 8L^3Q - 4L^2B,$$

leading on reduction to a cubic in L,

$$(3)\qquad QL^3 + BL^2 - AQL + Q^2 = 0,$$

which can be factorized into

$$(4)\qquad \left(L + \sqrt{\frac{x_2 x_3}{-x_1}}\right)\left(L - \sqrt{\frac{-x_1 x_3}{x_2}}\right)\left(L - \sqrt{\frac{-x_1 x_2}{x_3}}\right) = 0,$$

or

$$(5)\qquad (L - Q/x_1)(L - Q/x_2)(L - Q/x_3) = 0.$$

With

$$L' = L = -\sqrt{\frac{x_2 x_3}{-x_1}}, \qquad Q = Lx_1, \tag{6}$$

$$\mathrm{sh}\theta_1 = 0, \qquad z_1 = 1, \tag{7}$$

$$\tfrac{1}{2}M^2(1-z) = x - x_1, \quad M\sin\tfrac{1}{2}\theta = \sqrt{(x-x_1)}, \tag{8}$$

$$M^2 = L^2 - x_1 + x_2 + x_3 = (x_3 - x_1)(x_2 - x_1)/(-x_1); \tag{9}$$

and from (6) §11

$$M\cos\tfrac{1}{2}\theta\, e^{\chi i} = \frac{L\sqrt{(x-x_1)} + i\sqrt{(x_3-x)(x-x_2)}}{\sqrt{x}}, \tag{10}$$

$$M\cos\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{L\sqrt{(x-x_1)} + i\sqrt{(x_3-x)(x-x_2)}}{\sqrt{x}}\, e^{iI(v)} \tag{11}$$
$$= e^{iI(v-\omega_1)},$$

$$\cot\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{e^{iI(v-\omega_1)}}{\sqrt{(x-x_1)}}, \tag{12}$$

with

$$\frac{p}{n} = \frac{L - P(v)}{M} = \frac{P(\omega_1 - v)}{M}, \tag{13}$$

because

$$P(v) + P(\omega_1 - v) = -\sqrt{\frac{x_2 x_3}{-x_1}} = \frac{Q}{x_1}. \tag{14}$$

In this case the axis does not reach the highest vertical position, but describes an intermediate path.

But with

$$L' = L = \sqrt{\frac{-x_1 x_3}{x^2}}, \qquad Q = Lx_2, \tag{15}$$

$$\sin\theta_2 = 0, \qquad z_2 = 1, \tag{16}$$

$$\tfrac{1}{2}M^2(1-z) = x - x_2, \qquad M\sin\tfrac{1}{2}\theta = \sqrt{(x-x_2)}, \tag{17}$$

$$M^2 = L^2 + x_1 - x_2 + x_3 = (x_3 - x_2)(x_2 - x_1)/x_2, \tag{18}$$

$$M\cos\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{L\sqrt{(x-x_2)} + i\sqrt{(x_3-x)(x-x_1)}}{\sqrt{x}}\, e^{iI} = e^{iI(\omega_2 - v)}, \tag{19}$$

$$\cot\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{e^{iI(\omega_2 - v)}}{\sqrt{(x-x_2)}}, \tag{20}$$

$$\frac{p}{n} = \frac{P(\omega_2 - v)}{M}, \tag{21}$$

giving an upper rosette curve, in which the axis passes periodically through the highest vertical position.

With

$$(22)\qquad L' = -L = -\sqrt{\frac{-x_1x_2}{x_3}}, \quad Q = Lx_3,$$

$$(23)\qquad \sin\theta_3 = 0, \qquad z_3 = -1,$$

$$(24)\qquad \tfrac{1}{2}M^2(1+z) = x_3 - x, \qquad M\cos\tfrac{1}{2}\theta = \sqrt{(x_3 - x)},$$

$$(25)\qquad M^2 = -L^2 - x_1 - x_2 + x_3 = (x_3 - x_1)(x_3 - x_2)/x_3,$$

$$(26)\qquad M\sin\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{-L\sqrt{(x_3 - x)} + i\sqrt{(x - x_2)(x - x_1)}}{\sqrt{x}}\, e^{iI} = e^{iI(\omega_3 - v)},$$

$$(27)\qquad \tan\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{e^{iI(\omega_3 - v)}}{\sqrt{(x_3 - x)}},$$

$$(28)\qquad \frac{p}{n} = \frac{P(v - \omega_3)}{M},$$

giving a lower rosette curve, in which the axis passes periodically through the lowest vertical position.

Expressed in a tabular form we shall find for these rosettes:

	INTERMEDIATE	UPPER	LOWER
$\mathrm{ch}\,\tfrac{1}{2}\theta_1 =$	1	$\mathrm{nd}(1-f)K'$	$\mathrm{nd}(1-f)K'$
$\cos\tfrac{1}{2}\theta_2 =$	$\mathrm{dn}\,fK'$	1	$\mathrm{dn}\,fK'$
$\cos\tfrac{1}{2}\theta_3 =$	$\mathrm{cn}\,fK'$	$\mathrm{sn}\,fK'$	0
$\tfrac{1}{2}\theta_3 =$	$\mathrm{am}\,fK'$	$\tfrac{1}{2}\pi - \mathrm{am}\,fK'$	$\tfrac{1}{2}\pi$

Stereoscopic representations of an upper and lower rosette curve drawn for a parameter

$$(29)\qquad v = K + \tfrac{1}{2}K'i,$$

are shown in the diagram drawn by the late Mr. T. I. Dewar, published in *Engineering*, Oct. 10, 1897.

16. Cusps will exist on the circle $z = z_2$, if

$$(1)\qquad z_2 = \frac{G}{G'} = \frac{LM^2}{L^3 + AL - 2Q},$$

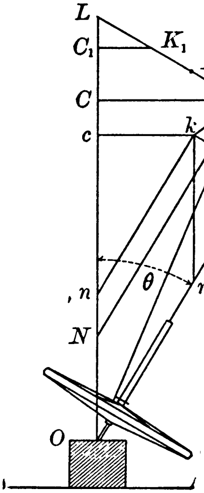

FIG. 1.

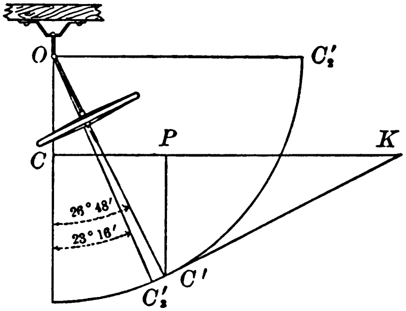

FIG. 3.

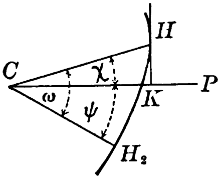

FIG. 2.

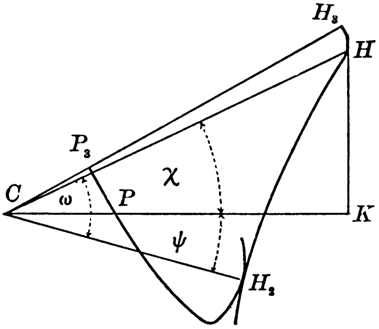

FIG. 4.

$$(2) \qquad (L^2 + x_1 - x_2 + x_3)(L^3 + AL - 2Q) = LM^4$$

$$L = (L^2 + A)^2 - 8L^2Q - 4LB,$$

reducing to

$$(3) \quad \left(L - \sqrt{\frac{-x_1x_3}{x_2}}\right)\left\{\left(L - \sqrt{\frac{-x_1x_3}{x_2}}\right)^2 - \frac{(x_3 - x_2)(x_2 - x_1)}{x_2}\right\} = 0,$$

of which the second factor must be taken, the first factor giving an upper rosette curve; and now for cusps

$$(4) \qquad \tfrac{1}{2}M^2 \sin\theta_2 = \sqrt{(x_3 - x_2)(x_2 - x_1)},$$

$$(5) \qquad \tfrac{1}{2}M^2 = \frac{(x_3 - x_2)\sqrt{(-x_1)(x_2 - x_1)} + (x_2 - x_1)\sqrt{x_3(x_3 - x_2)}}{x_2},$$

$$(6) \quad \sin\theta_2 = \frac{\sqrt{x_3(x_2 - x_1)} - \sqrt{(-x_1)(x_3 - x_2)},}{x_3 - x_1} = \kappa'(\mathrm{dn}fK' - \kappa\,\mathrm{sn}fK'),$$

$$(7) \quad \cos\theta_2 = \frac{\sqrt{x_3(x_3 - x_2)} + \sqrt{(-x_1)(x_2 - x_1)}}{x_3 - x_1} = \kappa\,\mathrm{dn}fK' + \kappa'^2\mathrm{sn}fK'.$$

When z_1 and z_2 have changed places, cusps can exist on the circle $z = z_1$, and

$$(8) \qquad \left(L + \sqrt{\frac{x_2x_3}{-x_1}}\right)^2 = \frac{(x_3 - x_1)(x_2 - x_1)}{-x_1},$$

with similar expressions for M^2 and θ_1;

$$(9) \qquad \tfrac{1}{2}M^2 = \sqrt{\frac{(x_3 - x_1)(x_2 - x_1)}{-x_1}}\left[\sqrt{x_2(x_3 - x_1)} - \sqrt{x_3(x_2 - x_1)}\right],$$

$$(10) \qquad \sin\theta_1 = \frac{\sqrt{x_2(x_3 - x_1)} + \sqrt{x_3(x_2 - x_1)}}{x_3 - x_2}.$$

To produce cusps experimentally the bicycle wheel after being spun is placed gently in the support *O* in fig. 1, or the axis in fig. 3 is let go without impulse. If let fall from a height into *O* a looped figure is produced, or if the axis receives an impulse in a vertical plane.

(*To be continued.*)